Class of bipartite quantum states satisfying the original Bell inequality

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Abstract

In a general setting, we introduce a new bipartite state property sufficient for the validity of the perfect correlation form of the original Bell inequality for any three bounded quantum observables. A bipartite quantum state with this property does not necessarily exhibit perfect correlations. The class of bipartite states specified by this property includes both separable and nonseparable states. We prove analytically that, for any dimension $d \geq 3$, every Werner state, separable or nonseparable, belongs to this class.

1 Introduction

The validity of Bell-type inequalities in the quantum case has been intensively discussed in the literature from the fundamental publications of J. S. Bell [1] and J. F. Clauser, M. A. Horne, A. Shimony and R. A. Holt [2]. At present, Bell-type inequalities are widely used in quantum information processing. However, from the pioneering paper of R. Werner [3] up to now a general structure of bipartite quantum states not violating Bell-type inequalities has not been well formalized. The recent results of M. Terhal, A. C. Doherty and D. Schwab [4] represent a significant progress in this direction but concern only the validity of CHSH[1]-form inequalities. Moreover, the proof in [4] of one of its main results on CHSH-form inequalities (see [4], theorem 2), specified for the case of discrete outcomes, cannot be explicitly extended to a general spectral case.

A general structure of bipartite quantum states satisfying the original Bell inequality for any three bounded quantum observables[2] has not been analyzed in the physical and mathematical literature.

The original derivation of the perfect correlation form of the Bell inequality in [1] is essentially based on the assumption of perfect correlations whenever one and the same quantum observable is measured on both "sides". However, for a bipartite quantum state, separable or nonseparable, the condition on perfect correlations cannot be fulfilled for every quantum

[1]Abbreviation of Clauser-Horne-Shimony-Holt.

[2]In classical probability, the product expectation values satisfy the perfect correlation form of the original Bell inequality for any three bounded classical observables and any classical state (see appendix of [5], for the proof). Recall that the original derivation of this inequality in [1] is true only for dichotomic classical observables with values $\pm\lambda$.

observable. On the other hand, as we proved in a general setting in [5], there exist[3] separable quantum states that satisfy the perfect correlation form of the original Bell inequality for any three bounded quantum observables and do not necessarily exhibit perfect correlations. From the mathematical point of view, the Bell correlation assumptions in [1] represent only sufficient but not necessary conditions for a bipartite quantum state to satisfy the original Bell inequality. Therefore, there must exist more general sufficient conditions.

In the present paper, we analyze the validity of the CHSH inequality and the original Bell inequality in a general bipartite quantum case. We introduce a new bipartite state property sufficient for the validity of the perfect correlation form of the original Bell inequality for any three bounded quantum observables. This state property is purely geometrical and is associated with the existence for a bipartite quantum state of a special type dilation[4] to an extended tensor product Hilbert space. Satisfying the perfect correlation form of the original Bell inequality for any three bounded quantum observables, a bipartite quantum state with this property does not necessarily exhibit perfect correlations.

We prove that every Werner state [3] on $\mathbb{C}^d \otimes \mathbb{C}^d$, $\forall d \geq 3$, separable or nonseparable, has this property and, therefore, satisfies the original Bell inequality for any three quantum observables on $\mathbb{C}^d$. In the two-qubit case, the original Bell inequality holds for any separable Werner state.

2 Source-operators and DSO states

For a quantum state ρ on a separable complex Hilbert space $\mathcal{H} \otimes \mathcal{H}$, possibly infinite dimensional, let us introduce self-adjoint trace class operators $T_{\blacktriangleright}$ and $T_{\blacktriangleleft}$ on $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$, defined by the relations

$$\mathrm{tr}_{\mathcal{H}}^{(2)}[T_{\blacktriangleright}] = \mathrm{tr}_{\mathcal{H}}^{(3)}[T_{\blacktriangleright}] = \rho, \qquad \mathrm{tr}_{\mathcal{H}}^{(1)}[T_{\blacktriangleleft}] = \mathrm{tr}_{\mathcal{H}}^{(2)}[T_{\blacktriangleleft}] = \rho. \tag{1}$$

Here, $\mathrm{tr}_{\mathcal{H}}^{(k)}[\cdot]$, $k = 1, 2, 3$, denotes the partial trace over the elements of $\mathcal{H}$ standing in the k-th place in $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$ and the lower indices of $T_{\blacktriangleright}$, $T_{\blacktriangleleft}$ indicate the direction of extension. Note that $T_{\blacktriangleright}$, $T_{\blacktriangleleft}$ are not necessarily positive[5].

For concreteness, we call any of dilations defined by (1) *a source-operator for a bipartite state* ρ. For any bipartite state ρ, source-operators $T_{\blacktriangleright}$, $T_{\blacktriangleleft}$ exist[6]. From (1) it follows that, for any source-operator T, its trace $\mathrm{tr}[T] = 1$. Since any positive source-operator is a density operator, we refer to it as a *density source-operator* or a *DSO*, for short.

The notion of a source-operator allows us to derive the following general upper bounds[7] for quantum product averages in an arbitrary bipartite state ρ:

$$\begin{aligned} \left| \mathrm{tr}[\rho(W_1 \otimes W_2)] - \mathrm{tr}[\rho(W_1 \otimes \widetilde{W}_2)] \right| &\leq ||T_{\blacktriangleright}||_1 \{1 - \mathrm{tr}[\sigma_{T_{\blacktriangleright}}^{(1)}(W_2 \otimes \widetilde{W}_2)] \}, && (2) \\ \left| \mathrm{tr}[\rho(W_1 \otimes W_2) - \mathrm{tr}[\rho(\widetilde{W}_1 \otimes W_2)] \right| &\leq ||T_{\blacktriangleleft}||_1 \{1 - \mathrm{tr}[\sigma_{T_{\blacktriangleleft}}^{(3)}(W_1 \otimes \widetilde{W}_1)] \}. && (3) \end{aligned}$$

Here: (i) W_1, $\widetilde{W}_1$, W_2, $\widetilde{W}_2$ are any bounded quantum observables on $\mathcal{H}$ with operator norms $|| \cdot || \leq 1$; (ii) $T_{\blacktriangleright}$, $T_{\blacktriangleleft}$ are any source-operators for a state ρ; (iii) $\|T\|_1 := \mathrm{tr}[|T|]$ is the trace

[3] See [5], section 3.B.1, Eq. (49).

[4] This dilation differs from dilations introduced in [4].

[5] They also do not necessarily have symmetries specified for dilations in [4].

[6] See [6] (sec. 2.1, proposition 1), for the proof.

[7] See [6] (sec. 2.2, proposition 3), for the proof.

norm of a source-operator T and $\sigma_T^{(j)} := \frac{1}{\|T\|_1}\mathrm{tr}_{\mathcal{H}}^{(j)}[|T|]$, $j = 1, 3$, is the density operator on $\mathcal{H} \otimes \mathcal{H}$ induced by T.

If, for a bipartite state ρ, there exists a density source-operator then we call this ρ *a density source-operator state*[8] or a *DSO state,* for short.

For a density source-operator R, its trace norm $\|R\|_1 = 1$. Therefore, for a DSO state ρ, the bounds (2), (3), specified with the corresponding density source-operators $R_{\blacktriangleright}$ or $R_{\blacktriangleleft}$, take the form

$$\left| \mathrm{tr}[\rho(W_1 \otimes W_2)] - \mathrm{tr}[\rho(W_1 \otimes \widetilde{W}_2)] \right| \leq 1 - \mathrm{tr}[\sigma_{R_{\blacktriangleright}}^{(1)}(W_2 \otimes \widetilde{W}_2)] \}, \tag{4}$$

$$\left| \mathrm{tr}[\rho(W_1 \otimes W_2)] - \mathrm{tr}[\rho(\widetilde{W}_1 \otimes W_2)] \right| \leq 1 - \mathrm{tr}[\sigma_{R_{\blacktriangleleft}}^{(3)}(W_1 \otimes \widetilde{W}_1)] \}, \tag{5}$$

where $\sigma_R^{(j)} = \mathrm{tr}_{\mathcal{H}}^{(j)}[R]$.

We introduce the following general statement on DSO states.

Theorem 1 *A DSO state ρ on $\mathcal{H} \otimes \mathcal{H}$ satisfies the original CHSH inequality [2]:*

$$\left| \mathrm{tr}[\rho(W_1 \otimes W_2 + W_1 \otimes \widetilde{W}_2 + \widetilde{W}_1 \otimes W_2 - \widetilde{W}_1 \otimes \widetilde{W}_2)] \right| \leq 2, \tag{6}$$

for any bounded quantum observables[9] W_1, $\widetilde{W}_1$, W_2, $\widetilde{W}_2$ *on* $\mathcal{H}$ *with operator norms* $\|\cdot\| \leq 1$.

Proof. Let a DSO state ρ have a density source-operator $R_{\blacktriangleright}$. Then, combining in the left-hand side of the inequality (6) the first term with the second while the third term with the fourth and applying further (4), we have

$$\begin{aligned} & \left| \mathrm{tr}[\rho(W_1 \otimes W_2 + W_1 \otimes \widetilde{W}_2 + \widetilde{W}_1 \otimes W_2 - \widetilde{W}_1 \otimes \widetilde{W}_2)] \right| \\ \leq & \left| \mathrm{tr}[\rho(W_1 \otimes W_2 + W_1 \otimes \widetilde{W}_2)] \right| + \left| \mathrm{tr}[\rho(\widetilde{W}_1 \otimes W_2 - \widetilde{W}_1 \otimes \widetilde{W}_2)] \right| \leq 2. \end{aligned} \tag{7}$$

If a state ρ has a density source-operator $R_{\blacktriangleleft}$, then we prove (6) quite similarly - by combining in the left-hand side of (6) the first term with the third while the second term with the fourth and applying further (5). ■

Any separable state is a DSO state[10]. In section 3.1, we present examples of nonseparable DSO states. Notice that a nonseparable DSO state does not necessarily admit a local hidden variable model in the sense formulated in [3].

Consider now a generalized joint quantum measurement, with real-valued outcomes $\lambda_1, \lambda_2 \in \Lambda \subseteq [-1, 1]$ of any spectral type and performed on a bipartite state ρ. Let, under this measurement, the joint probability that outcomes λ_1 and λ_2 belong to subsets B_1, $B_2 \subseteq \Lambda$, respectively, have the form[11]: $\mathrm{tr}[\rho(M_1^{(a)}(B_1) \otimes M_2^{(b)}(B_2))]$, where $M_1^{(a)}$ and $M_2^{(b)}$ are positive operator-valued (POV) measures of both parties involved and parameters a, b specify measurement settings of these parties. In the physical literature, this type of a joint measurement

[8] Any bipartite state that has an (s_a, s_b)-symmetric extension (according to the terminology used in [4]) represents a DSO state. From the other side, the corresponding symmetrization of a density source-operator results in an (1,2) or (2,1) symmetric extension.

[9] In case of an infinite dimensional $\mathcal{H}$, observables may be of any spectral type.

[10] Let $\sum_i \alpha_i \rho_i \otimes \widetilde{\rho}_i$, $\alpha_i > 0$, $\sum_i \alpha_i = 1$, be a separable representation of a separable state. Then, for this state, $\sum_i \alpha_i \rho_i \otimes \widetilde{\rho}_i \otimes \widetilde{\rho}_i$ and $\sum_i \alpha_i \rho_i \otimes \rho_i \otimes \widetilde{\rho}_i$ represent density source-operators.

[11] See, for example, in [5].

is usually associated with Alice and Bob names. Under an Alice/Bob joint measurement, specified by a pair[12] (a, b) of measurement settings, the expectation value $\langle\lambda_1\lambda_2\rangle_\rho^{(a,b)}$ of the product $\lambda_1\lambda_2$ of outcomes has the form

$$\langle\lambda_1\lambda_2\rangle_\rho^{(a,b)} := \int_{\Lambda\times\Lambda} \lambda_1\lambda_2 \mathrm{tr}[\rho(M_1^{(a)}(d\lambda_1)\otimes M_2^{(b)}(d\lambda_2))] = \mathrm{tr}[\rho(W_1^{(a)}\otimes W_2^{(b)})], \tag{8}$$

where $W_1^{(a)} := \int_\Lambda \lambda_1 M_1^{(a)}(d\lambda_1)$ and $W_2^{(b)} := \int_\Lambda \lambda_2 M_2^{(b)}(d\lambda_2)$ are bounded quantum observables, representing on $\mathcal{H}$ marginal measurements of Alice and Bob, respectively.

Due to theorem 1 and the representation (8), we immediately derive that a DSO state ρ satisfies the CHSH inequality:

$$\left|\langle\lambda_1\lambda_2\rangle_\rho^{(a_1,b_1)} + \langle\lambda_1\lambda_2\rangle_\rho^{(a_1,b_2)} + \langle\lambda_1\lambda_2\rangle_\rho^{(a_2,b_1)} - \langle\lambda_1\lambda_2\rangle_\rho^{(a_2,b_2)}\right| \leq 2, \tag{9}$$

under any generalized Alice/Bob joint measurements with outcomes $|\lambda_1|$, $|\lambda_2| \leq 1$ of an arbitrary spectral type.

3 Bell class

Let a DSO state ρ on $\mathcal{H}\otimes\mathcal{H}$ have a density source-operator R with the special dilation property

$$\mathrm{tr}_{\mathcal{H}}^{(1)}[R] = \mathrm{tr}_{\mathcal{H}}^{(2)}[R] = \mathrm{tr}_{\mathcal{H}}^{(3)}[R] = \rho. \tag{10}$$

This is, in particular, the case where a state ρ is reduced from a symmetric density operator on $\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}$.

Theorem 2 *If a DSO state ρ on $\mathcal{H}\otimes\mathcal{H}$ has a density source-operator with the property (10) then this DSO state ρ satisfies the perfect correlation form of the original Bell inequality [1]:*

$$\begin{aligned} \left|\ \mathrm{tr}[\rho(W_1\otimes W_2)] - \mathrm{tr}[\rho(W_1\otimes \widetilde{W}_2)]\ \right| &\leq 1 - \mathrm{tr}[\rho(W_2\otimes\widetilde{W}_2)], \\ \left|\ \mathrm{tr}[\rho(W_1\otimes W_2)] - \mathrm{tr}[\rho(\widetilde{W}_1\otimes W_2)]\ \right| &\leq 1 - \mathrm{tr}[\rho(W_1\otimes\widetilde{W}_1)], \end{aligned} \tag{11}$$

for any bounded quantum observables W_1, $\widetilde{W}_1$, W_2, $\widetilde{W}_2$ on $\mathcal{H}$ with operator norms $\|\cdot\| \leq 1$.

Proof. In the inequalities (4), (5), specified for a Bell class DSO state ρ, let us take a DSO R with the property (10). Then, due to (10), $\sigma_R^{(1)} = \sigma_R^{(3)} = \rho$, and, therefore, the inequalities (4), (5) reduce to (11). ■

In view of theorem 2, we refer to a DSO state having a density source-operator with the special dilation property (10) as *a DSO state of the Bell class.*

The set of all Bell class DSO states on $\mathcal{H}\otimes\mathcal{H}$ is convex and includes both separable and nonseparable states. It is easy to verify that a separable state of the special form[13]: $\sum_m \xi_m\rho_m\otimes\rho_m$, where $\xi_m > 0$, $\sum_m \xi_m = 1$, represents a Bell class DSO state. In section 3.1, we present examples of nonseparable Bell class DSO states.

[12] Here, the first argument in a pair refers to a marginal measurement (say of Alice) with outcomes λ_1, while the second argument - to a Bob marginal measurement, with outcomes λ_2.

[13] Proved by us in [5] (sec. 3.B.1) to satisfy (11) for any three bounded quantum observables.

Note that, satisfying the perfect correlation form of the original Bell inequality for any three bounded quantum observables on $\mathcal{H}$, *a Bell class DSO state (separable or nonseparable) does not necessarily exhibit the perfect correlation of outcomes if one and the same quantum observable is measured on both "sides"*. In case of a dichotomic quantum observable W_2, with eigenvalues ± 1, the latter means that a Bell class DSO state ρ satisfies the first inequality in (11) even if, for this state, the correlation function $\mathrm{tr}[\rho(W_2 \otimes W_2)] \neq 1$.

From theorem 2 and the representation (8) it follows that, under generalized Alice/Bob joint quantum measurements, a Bell class DSO state ρ satisfies the relation

$$\begin{aligned} \left| \langle \lambda_1 \lambda_2 \rangle_\rho^{(a,b_1)} - \langle \lambda_1 \lambda_2 \rangle_\rho^{(a,b_2)} \right| &= \left| \mathrm{tr}[\rho(W_1^{(a)} \otimes W_2^{(b_1)})] - \mathrm{tr}[\rho(W_1^{(a)} \otimes W_2^{(b_2)})] \right| \\ &\leq 1 - \mathrm{tr}[\rho(W_2^{(b_1)} \otimes W_2^{(b_2)})]. \end{aligned} \tag{12}$$

This relation implies that, under generalized Alice/Bob joint measurements, a Bell class DSO state ρ satisfies the perfect correlation form of the original Bell inequality:

$$\left| \langle \lambda_1 \lambda_2 \rangle_\rho^{(a,b_1)} - \langle \lambda_1 \lambda_2 \rangle_\rho^{(a,b_2)} \right| \leq 1 - \langle \lambda_1 \lambda_2 \rangle_\rho^{(b_1,b_2)}, \tag{13}$$

whenever $W_2^{(b_1)} = W_1^{(b_1)}$, that is, if POV measures of Alice and Bob satisfy the condition $\int_\Lambda \lambda_1 M_1^{(b_1)}(d\lambda_1) = \int_\Lambda \lambda_2 M_2^{(b_1)}(d\lambda_2)$.

The latter operator condition on POV measures does not imply the perfect correlation of outcomes and is always satisfied under Alice and Bob projective measurements of one and the same quantum observable on both "sides".

3.1 Examples

Consider on $\mathbb{C}^d \otimes \mathbb{C}^d$, $d \geq 2$, Werner states [3]: $\rho^{(d,\Phi)} = \frac{d-\Phi}{d^3-d} I_{\mathbb{C}^d \otimes \mathbb{C}^d} + \frac{d\Phi - 1}{d^3-d} V_d$, $\forall \Phi \in [-1,1]$, widely used in quantum information processing. Represented otherwise, Werner states have the form

$$\rho^{(d,\Phi)} = \frac{1+\Phi}{2} \frac{P_d^{(+)}}{r_d^{(+)}} + \frac{1-\Phi}{2} \frac{P_d^{(-)}}{r_d^{(-)}}. \tag{14}$$

Here: (i) $V_d(\psi_1 \otimes \psi_2) := \psi_2 \otimes \psi_1$ is the permutation (flip) operator on $\mathbb{C}^d \otimes \mathbb{C}^d$; (ii) $P_d^{(\pm)} = \frac{1}{2}(I_{\mathbb{C}^d \otimes \mathbb{C}^d} \pm V_d)$ are the orthogonal projections onto the symmetric (plus sign) and antisymmetric (minus sign) subspaces of $\mathbb{C}^d \otimes \mathbb{C}^d$ with dimensions $r_d^{(\pm)} = \mathrm{tr}[P_d^{(\pm)}] = \frac{d(d\pm 1)}{2}$. For any $d \geq 2$, a Werner state $\rho^{(d,\Phi)}$ is separable if $\Phi \in [0,1]$ and nonseparable otherwise.

Theorem 3 *(a) For a dimension $d \geq 3$, every Werner state $\rho^{(d,\Phi)}$, separable or nonseparable, is a Bell class DSO state. (b) A two-qubit Werner state $\rho^{(2,\Phi)}$ is a Bell class DSO state whenever $\Phi \in [0,1]$, that is, if $\rho^{(2,\Phi)}$ is separable.*

Proof. Introduce on $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$, $\forall d \geq 2$, the orthogonal projections[14]:

$$\begin{aligned} Q_d^{(\pm)}(\psi_1 \otimes \psi_2 \otimes \psi_3) \ : &= \frac{1}{6}\{\psi_1 \otimes \psi_2 \otimes \psi_3 \pm \psi_2 \otimes \psi_1 \otimes \psi_3 \pm \psi_1 \otimes \psi_3 \otimes \psi_2 \\ &\pm \psi_3 \otimes \psi_2 \otimes \psi_1 + \psi_2 \otimes \psi_3 \otimes \psi_1 + \psi_3 \otimes \psi_1 \otimes \psi_2\}, \end{aligned} \tag{15}$$

[14] $Q_d^{(+)}$ is the projection on the symmetric subspace of $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$.

$\forall \psi_1, \psi_2, \psi_3 \in \mathbb{C}^d$. These projections have the form

$$\begin{aligned} 6Q_d^{(\pm)} &= I_{\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d} \pm V_d \otimes I_{\mathbb{C}^d} \pm I_{\mathbb{C}^d} \otimes V_d \pm (I_{\mathbb{C}^d} \otimes V_d)(V_d \otimes I_{\mathbb{C}^d})(I_{\mathbb{C}^d} \otimes V_d) \\ &\quad + (I_{\mathbb{C}^d} \otimes V_d)(V_d \otimes I_{\mathbb{C}^d}) + (V_d \otimes I_{\mathbb{C}^d})(I_{\mathbb{C}^d} \otimes V_d). \end{aligned} \tag{16}$$

Taking into account that[15]

$$\begin{aligned} I_{\mathbb{C}^d \otimes \mathbb{C}^d} &= \mathrm{tr}_{\mathbb{C}^d}^{(j)}[V_d \otimes I_{\mathbb{C}^d}] = \mathrm{tr}_{\mathbb{C}^d}^{(k)}[I_{\mathbb{C}^d} \otimes V_d] = \mathrm{tr}_{\mathbb{C}^d}^{(m)}[(I_{\mathbb{C}^d} \otimes V_d)(V_d \otimes I_{\mathbb{C}^d})(I_{\mathbb{C}^d} \otimes V_d)], \\ \forall\, j &= 1,2, \quad \forall k = 2,3, \quad \forall m = 1,3; \\ V_d &= \frac{1}{d}\mathrm{tr}_{\mathbb{C}^d}^{(3)}[V_d \otimes I_{\mathbb{C}^d}] = \frac{1}{d}\mathrm{tr}_{\mathbb{C}^d}^{(1)}[I_{\mathbb{C}^d} \otimes V_d] = \frac{1}{d}\mathrm{tr}_{\mathbb{C}^d}^{(2)}[(I_{\mathbb{C}^d} \otimes V_d)(V_d \otimes I_{\mathbb{C}^d})(I_{\mathbb{C}^d} \otimes V_d)] \\ &= \mathrm{tr}_{\mathbb{C}^d}^{(n)}[(I_{\mathbb{C}^d} \otimes V_d)(V_d \otimes I_{\mathbb{C}^d})] = \mathrm{tr}_{\mathbb{C}^d}^{(n)}[(V_d \otimes I_{\mathbb{C}^d})(I_{\mathbb{C}^d} \otimes V_d)], \quad \forall n = 1,2,3, \end{aligned} \tag{17}$$

we derive

$$\mathrm{tr}_{\mathbb{C}^d}^{(j)}[Q_d^{(\pm)}] = \frac{d \pm 2}{6}(I_{\mathbb{C}^d \otimes \mathbb{C}^d} \pm V_d) = \frac{d \pm 2}{3} P_d^{(\pm)}, \quad \forall j = 1,2,3, \tag{18}$$

and $\mathrm{tr}[Q_d^{(\pm)}] = \frac{d(d\pm1)(d\pm2)}{6}$.

Consider on $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$, $\forall d \geq 3$, the density operator

$$R^{(d,\Phi)} = \frac{1+\Phi}{2}\frac{6Q_d^{(+)}}{d(d+1)(d+2)} + \frac{1-\Phi}{2}\frac{6Q_d^{(-)}}{d(d-1)(d-2)}. \tag{19}$$

Due to (14), (18), $\mathrm{tr}_{\mathbb{C}^d}^{(j)}[R^{(d,\Phi)}] = \rho^{(d,\Phi)}$, for any $j = 1,2,3$. Therefore, for any state $\rho^{(d,\Phi)}$, with $d \geq 3$ and $\Phi \in [-1,1]$, the operator $R^{(d,\Phi)}$ is a density source-operator (DSO) with the property (10). This proves statement (a).

For a state $\rho^{(2,\Phi)} = \frac{1-\Phi}{2} I_{\mathbb{C}^2 \otimes \mathbb{C}^2} + \frac{2\Phi-1}{3} P_2^{(+)}$, consider the operator

$$\widetilde{R}^{(2,\Phi)} = \frac{1-\Phi}{4} I_{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2} + \frac{2\Phi - 1}{4} Q_2^{(+)}. \tag{20}$$

This operator satisfies the relation $\mathrm{tr}_{\mathbb{C}^d}^{(j)}[\widetilde{R}^{(2,\Phi)}] = \rho^{(2,\Phi)}$, for any $j = 1,2,3$, and is positive whenever $\Phi \in [0,1]$. Hence, for any state $\rho^{(2,\Phi)}$, with $\Phi \in [0,1]$, the operator $\widetilde{R}^{(2,\Phi)}$ represents a DSO with the property (10). The latter proves statement (b). ■

In view of theorems 2, 3, for any dimension $d \geq 3$, every Werner state, separable or nonseparable, satisfies the perfect correlation form of the original Bell inequality for any three quantum observables on $\mathbb{C}^d$. For $d = 2$, the original Bell inequality holds for any separable Werner state.

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[15] Here, we use: $V_d^2 = I_{\mathbb{C}^d \otimes \mathbb{C}^d}$, $\mathrm{tr}_{\mathbb{C}^d}^{(j)}[V_d] = I_{\mathbb{C}^d}$, $\mathrm{tr}_{\mathbb{C}^d}^{(j)}[P_d^{(\pm)}] = \frac{d\pm1}{2} I_{\mathbb{C}^d}$, $\forall\, j = 1,2$.

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